Ergodic Theory - Week 7

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1 Classifying measure preserving systems

P1. Show that any factor of an ergodic system is ergodic. Find an example of a non-ergodic system with an ergodic factor.

> Let $(X, \mathcal{A}, \mu, T)$ be an ergodic measure preserving system, let $Y, \mathcal{B}, \nu, S)$ be a factor, and let $\pi : X \to Y$ be the associated factor map. Let f be a S-invariant function. Then we have that
>
> $$f \circ \pi = f \circ S \circ \pi = f \circ \pi \circ T \quad \mu - \text{almost everywhere.}$$
>
> Since T is ergodic, it follows that $f \circ \pi$ is almost everywhere constant, and thus, f is almost everywhere constant.
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> Let $a \in [0, 1)$be irrational. The system $(\mathbb{T}^2, \mathcal{B}_{\mathbb{T}^2}, m_{\mathbb{T}^2}, R_a \times R_a)$ is not ergodic, but the rotation by a is an ergodic factor of this system (the associated map is the projection on the first coordinate). To see that this system is not ergodic, consider the function f given by $f(x, y) = x - y$, and notice that this is a non-constant invariant function.

P2. Let G be a compact abelian group. Show that a group rotation by some $\alpha \in G$ is ergodic if and only if $\{n\alpha\}_{n\in\mathbb{Z}}$ is a dense subgroup of G.
Hint: You may use that any function in $L^2(G)$ has a Fourier expansion $f = \sum_\chi \widetilde{f}(\chi)\chi$ where convergence is understood to be in the $L^2(G)$-norm.

> $(\Longrightarrow)$ Suppose the rotation T by $\alpha \in G$ is not dense in $\{n\alpha\}_{n\in\mathbb{Z}}$. Then there exists a non-trivial character ξ such that $\xi(n\alpha) = 1$ for all $n \in \mathbb{Z}$. Indeed, consider $H = \overline{\{n\alpha\}_{n\in\mathbb{Z}}}$ which is a compact strict subgroup of G. Consider the compact abelian group G/H and take any non-trivial character $\tilde{\xi}$ in G/H. Then, define $\xi(g) = \tilde{\xi}(g + H)$. Clearly $\xi(g) = 1$ for every $g \in H$, ξ is a morphism from G to $\mathbb{S}^1$, and it is continuous by being composition of continuous functions.
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> Now take $f = \xi$. Then f is a non-constant, T-invariant function, and therefore T is not ergodic.
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> $(\Longleftarrow)$ If $\{n\alpha\}_{n\in\mathbb{Z}}$ is dense in G, then for any non-trivial character ξ, we have $\xi(\alpha) \neq 1$. Let f be a T-invariant function in $L^2(G)$ and write its Fourier expansion
>
> $$f = \sum_\xi c_\xi \xi.$$
>
> Since $f \circ T = f$, we have
>
> $$\sum_\xi c_\xi \xi(\alpha)\xi = \sum_\xi c_\xi \xi.$$
>
> By the uniqueness of the Fourier coefficients, we have $c_\xi \xi(\alpha) = c_\xi$ for all ξ. Since $\xi(\alpha) \neq 1$ for

all non-trivial ξ, it follows that $c_\xi = 0$ for all $\xi \neq 1$. Then $f \equiv c_1$ which is constant, and this shows that T is ergodic.

P3. We call a system $(X, \mathcal{B}, \mu, T)$ totally ergodic if for every $k \in \mathbb{N}$ the map T^k is ergodic with respect to μ.

(a) Show that if $(X, \mathcal{B}, \mu, T)$ is totally ergodic, then for any $a, b \in \mathbb{N}$ we have

$$\operatorname*{lím}_{N \to +\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} T^{an+b} f - \int f \, d\mu \right\|_{L^2(\mu)} = 0.$$

This is a simple application of von Neumann's ergodic theorem. Indeed, observe that $T^{an+b} f = (T^a)^n (T^b f)$ and since the transformation T^a is ergodic, we deduce that (convergence here is in L^2-norm)

$$\frac{1}{N} \sum_{n=1}^{N} T^{an+b} f \to \int T^b f \, d\mu = \int f \, d\mu,$$

where the last equality follows from the fact that T preserves the measure μ.

(b) Show that an ergodic system is totally ergodic if and only if every eigenfunction with corresponding eigenvalue that is a root of unity is constant almost everywhere.

Assume that a system is totally ergodic and let f be an eigenfunction with eigenvalue λ that is a root of unity. Thus, there exists $r \in \mathbb{N}$ such that $\lambda^r = 1$. We conclude that

$$U_T^r f = U_T^{r-1} (\lambda f) = \cdots = \lambda^r f = f.$$

Since T^r is ergodic, we conclude that f is constant almost everywhere.

For the converse direction, assume that the system is not totally ergodic. Therefore, there exists $k \in \mathbb{N}$ and a non-constant function $f \in L^2(X)$ for which $U_T^k f = f$.

If $k = 1$, we get that $U_T f = f$, which means that f is an eigenfunction (with eigenvalue 1). Therefore, f is constant almost everywhere, which is a contradiction. Therefore, we get $k \geq 2$ and that T is ergodic.

Now, we suppose that $k \geq 2$. We consider the non-trivial roots of unity $\omega_i = e(\frac{i}{k})$ where $0 \leq i \leq k-1$ and define the functions

$$g_i = f + \omega_i U_T f + \cdots + \omega_i^{k-1} U_T^{k-1} f.$$

Observe that, since $U_T^k f = f$, we have

$$U_T g_i = U_T f + \omega_i U_T^2 f + \cdots + \omega_i^{k-1} f \implies U_T g_i = \frac{1}{\omega_i} g_i.$$

Therefore, g_i is an eigenfunction with eigenvalue a root of unity and our hypothesis implies that g_i is constant. In fact, for $i \neq 0$ (that is, $\omega_i \neq 1$), we get that since $U_T g_i = \bar{\omega}_i g_i$, the corresponding constant is zero and, hence, $g_i = 0$ almost everywhere.

We sum the functions g_i for $0 \leq i \leq k-1$ and deduce that

$$\sum_{i=0}^{k-1} \sum_{r=0}^{k-1} e\left(\frac{ri}{k}\right) U_T^r f$$

is constant almost everywhere. Changing the order of summation, we have

$$\sum_{r=0}^{k-1} U_T^r f \left(\sum_{i=0}^{k-1} e\left(\frac{ri}{k}\right) \right) = C \quad m-\text{a.e.}$$

for some constant C. Notice that

$$\sum_{i=0}^{k-1} e\left(\frac{ri}{k}\right) = \begin{cases} k, & \textbf{if } r = 0 \\ 0, & \textbf{if } r \neq 0 \end{cases} .$$

Namely, all the terms in the last sum disappear, apart from the term with $r = 0$. We conclude that $kf = C$ almost everywhere and thus f is a constant function, which is a contradiction. We conclude that the system is totally ergodic.

P4. **(a)** Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system, and let $\alpha \in (0,1)$ such that $e(\alpha)$ is an eigenvalue. Show that there exists a non-trivial group rotation that is a factor of $(X, \mathcal{B}, \mu, T)$.

Hint: When $a = r/q$ is rational (with q minimal among all such rational eigenvalues), construct a T^q-invariant set B such that $\mu(B) = 1/q$.

Let $f \in L^2(X)$ be the eigenfunction corresponding to the given eigenvalue. Then $Tf = e(\alpha)f$ and f is non-constant.

Suppose first that $\alpha \in \mathbb{Q}$. We write $\alpha = r/q$ for some $0 < r < q$ with $(r,q) = 1$. Then $T^q f = f$ and thus the transformation T^q is not ergodic. Let $q \geq 2$ be the minimal integer (larger than 1) for which T^q is not ergodic and consider $A \in \mathcal{B}$ such that $T^{-q}A = A$ and $\mu(A) \in (0,1)$.

Define $g = \sum_{n=0}^{q-1} T^n \mathbb{1}_A$ and notice that g is T-invariant. It follows by ergodicity that g is almost everywhere equal to some integer constant, and then $q\mu(A) = \int g d\mu \in \mathbb{Z}$. Thus, $\mu(A) = k/q$ for some $k \in \{1, \ldots, q-1\}$.

Claim. There exists a T^q-invariant set $B \in \mathcal{B}$ such that $\mu(B) = 1/q$.

To prove the claim, suppose that $k > 1$, since otherwise the claim holds trivially. First, we notice that by the pigeonhole principle, there exists some $m \in \{1, \ldots, q-1\}$ such that $\mu(A_1) > 0$, where $A_1 = A \cap T^{-m}A$. If $\mu(A_1) = \mu(A)$, then $T^{-m}A = A$ almost everywhere, which contradicts the minimality of q. Then $0 < \mu(A_1) < \mu(A) = k/q$. We can define g_1 as we defined g substituting A with A_1, and since A_1 is also T^q-invariant, then as before, there exists some integer $0 < k_1 < k$ such that $\mu(A_1) = k_1/q$. If $k_1 = 1$, then the claim follows by taking $B = A_1$. Otherwise we repeat the same argument for A_1 instead of A, to find some set A_2 with $\mu(A_2) = k_2/q$ for some $0 < k_2 < k_1$. Inductively, we find $j \in \{1, \ldots, q-1\}$ such that $B = A_j$ satisfies the claim.

Now, consider the set $Y = \{0, \ldots, q-1\}$, let $\mathcal{B}_Y$ be the discrete σ-algebra on Y, ν be the normalized counting measure on Y, and let $S : y \mapsto y + 1 \pmod{q}$. Then $(Y, \mathcal{B}_Y, \nu, S)$ is a measure-preserving system (a rotation on finitely many points). Using our claim, we deduce that the sets $B, T^{-1}B, \ldots, T^{q-1}B$ form a partition of X, thus for any $x \in X$, there exists a unique $y_x \in Y$ such that $x \in T^{-y_x}B$. We can then define the map $\pi : X \to Y$ by $\pi(x) = y_x$, and it is not hard to check that this is a factor map.

Now suppose that $\alpha \notin \mathbb{Q}$. We show that $(\mathbb{T}, \mathcal{B}_{\mathbb{T}}, m_{\mathbb{T}}, R_\alpha)$ is a factor of $(X, \mathcal{B}, \mu, T)$. We identify $\mathbb{T}$ with $[0,1)$ for simplicity. Let $\pi : X \to \mathbb{T}$ be the map defined uniquely by the equality $f(x) = e(\pi(x))$. Observe that this is well-defined for almost all $x \in X$, since $|f(x)| = 1$ almost everywhere.

We observe that

$$e(\pi(Tx)) = f(Tx) = e(\alpha)f(x) = e(\pi(x) + \alpha) = e(R_\alpha \pi(x)),$$

which means that $\pi(Tx) = R_a\pi(x)$ almost everywhere. Finally, π preserves $m_{\mathbb{T}}$. To see this, define a measure ν on $\mathbb{T}$ by $\nu(A) = \mu(\pi^{-1}A)$, $A \in \mathcal{B}$. We easily check that ν is R_α-invariant, and so it is the Haar measure $m_{\mathbb{T}}$ (the only measure invariant under an irrational rotation is the Haar measure).

(b) Show that given any countable subgroup $K \leq \mathbb{S}^1$, there exists a measure preserving system $(X, \mathcal{B}, \mu, T)$ on a Borel probability space such that K is the point-spectrum of T.
Hint: To construct the system take $X = \hat{K}$, $\mu = m_X$ the normalized Haar measure on X, and T to be some appropriate group rotation.

Give K the discrete topology, so that the dual group $\hat{K}$ is a compact metric abelian group. Following the hint, we take $X = \hat{K}$, $\mu = m_X$ the normalized Haar measure on X. Define the character $\theta \in X$ by $\theta(k) = k$, for $k \in K$. Define the transformation $T : X \to X$ to be the rotation by θ. In this way, $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system. We need to show that such a system has point-spectrum K.

First, we see that $K \leq spec(T)$. Indeed, we recall that Pontryagin's Theorem yields that K is isomorphic to $\hat{X}$ given by $k \to f_k$ where $f_k(x) = x(k)$. Therefore, for any character $f_k \in \hat{X}$ and $x \in X$

$$(U_T f_k)(x) = f_k(\theta x) = f_k(\theta)f_k(x) = \theta(k)f_k(x) = kf_k(x).$$

Thus f_k is an eigenfunction of U_T with eigenvalue k, and hence $K \leq spec(T)$.

To prove the other direction, recall that for any compact abelian group G we have that $\hat{G}$ is a orthonormal basis for $L^2(G)$. This means that if f is an eigenfunction of U_T with eigenvalue λ, then

$$f = \sum_{k \in K} c_k f_k,$$

where $c_k \in \mathbb{C}$ and the equality is in $L^2(X)$. Applying U_T and using the uniqueness of the coefficients, we have that

$$kc_k = \lambda c_k,$$

for every $k \in K$. Thus $c_k = 0$ unless $k = \lambda$, and therefore $f = c_\lambda f_\lambda$. Therefore $spec(T) \leq K$ since K is isomorphic to $\hat{X}$.